Dominant, weakly stable, uncovered sets: properties and extensions

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2008

ГОСУДАРСТВЕННЫЙ УНИВЕРСИТЕТ
ВЫСШАЯ ШКОЛА ЭКОНОМИКИ

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DOMINANT, WEAKLY STABLE, UNCOVERED SETS: PROPERTIES AND EXTENSIONS

Препринт WP7/2008/03
Серия WP7
Теория и практика
общественного выбора

Москва
ГУ ВШЭ
2008

УДК
ББК
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Редакторы серии WP7
«Теория и практика общественного выбора»
Ф.Т. Алескеров, Р.М. Нуреев

S **Subochev A. Dominant, Weakly Stable, Uncovered Sets: Properties and Extensions**[*]: Working paper WP7/2008/03. — Moscow: State University — Higher School of Economics, 2008. — 32 p.

УДК
ББК

Субочев А.Н. Доминирующее, слабоустойчивое и непокрытое множества: свойства и обобщения: Препринт WP7/2008/03. — М.: Изд. дом ГУ ВШЭ, 2008. — 32 с. (in English).

Ключевой проблемой моделирования коллективного выбора является то, что победитель Кондорсе, т.е. альтернатива более предпочтительная для коллектива, чем любая другая альтернатива при парном сравнении, в общем случае отсутствует. Поэтому с конца 70-х гг. прошлого века предпринимались попытки локализовать результат выбора в некотором всегда непустом подмножестве множества альтернатив, на котором определено отношение мажоритарного доминирования, играющее роль системы коллективных предпочтений.

Предметом данной работы является сравнительный анализ основных концепций, старых и новых, предлагавшихся в качестве решений задачи коллективного выбора. Сравниваются двенадцать множеств, построенных с помощью отношения мажоритарного доминирования: ядро, пять версий непокрытого множества, две версии минимального слабоустойчивого множества, незахваченное множество, незапертое множество, минимальное недоминируемое множество и минимальное доминирующее множество.

Основные результаты исследования, излагающиеся в работе, таковы.

I. Локализовано определение минимального слабоустойчивого множества, то есть сформулирован критерий, определяющий принадлежность альтернативы объединению минимальных слабоустойчивых множеств. С помощью этого критерия выявлена связь объединения минимальных слабоустойчивых множеств с отношением покрытия и с непокрытым множеством.

II. Для всех рассматриваемых множеств установлено наличие или отсутствие отношения включения как в общем случае, так и для такого важного частного случая, как турниры, то есть для таких случаев, когда отношение мажоритарного доминирования на всей совокупности альтернатив представимо полным, связным, асимметричным графом.

III. Для турниров на основе понятия устойчивости альтернативы и множества альтернатив построены обобщения непокрытого множества и слабоустойчивого множества – классы *k*-устойчивых альтернатив и *k*-устойчивых множеств. Установлено наличие отношения включения для этих классов.

IV. Построены обобщения минимального доминирующего множества и с их помощью выяснено, как устроена система доминирующих множеств в общем случае. Показано, что для турниров иерархии классов *k*-устойчивых альтернатив и *k*-устойчивых множеств в совокупности с иерархией доминирующих множеств порождают соответственно микро- и макроструктуру множества альтернатив, в основе которых лежит различие в устойчивости.

Методологической парадигмой исследования является теория рационального выбора. Основные методы и средства относятся к математическому аппарату теории графов и теории множеств.

[*]The author gratefully acknowledges invaluable support of Fuad Aleskerov (State University — Higher School of Economics (Moscow)), whose ideas, comments and advices have profoundly influenced the analysis presented here. The work was partially supported by Scientific Foundation of SU-HSE (grant №08-04-0008)

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1. Introduction

In most cases a Condorcet winner, i.e. an alternative more preferable for the majority of actors than any other alternative, does not exist and a core (a set of all alternatives undominated in majority relation) is empty. Therefore various sets of alternatives were proposed as solution concepts for majority voting games and/or as social choice rules. Below several such concepts are considered and compared: dominant set (Ward, 1961; Smith, 1973), minimal dominant set (Fishburn, 1977; Miller, 1977), undominated set (Ward, 1961), minimal undominated set (Schwartz, 1970, 1972), weakly stable and minimal weakly stable sets (Aleskerov, Kurbanov, 1999), uncovered set (Fishburn, 1977; Miller, 1980), uncaptured and untrapped sets (Duggan, 2007). In addition we introduce a new solution concept — k-stable sets of alternatives — and analyze its relations with other solution concepts listed above.

The structure of the article is as follows. In Section 2 the definitions of sets are given and their relationships are explored. In addition, a criterion to determine whether an alternative belongs to a minimal weakly stable set is established. It is shown that for tournaments an uncovered set is always a subset of a union of minimal weakly stable sets. It is also demonstrated that a hierarchy of dominant sets defines a "macrostructure" of majority relation.

In Section 3 the concept of stability is employed to generalize the notions of weakly stable and uncovered sets. The concepts of k-stable alternatives and sets are introduced and their properties and mutual relations are explored.

In Section 4 the results are summarized and interpreted. Table 2 and Table 3 summarize the relations between all sets discussed in the paper for general case and for tournaments respectively. It is also shown that the hierarchies of the classes of k-stable alternatives and k-stable sets combined with the system of dominant sets constitute a tournament structure based on different degrees of stability. Appendix 1 provides Examples and proofs of some propositions from the previous Sections. An algorithm for calculating the minimal dominant sets and the classes of k-stable alternatives is given in Appendix 2. Almost all proofs of Lemmas and Theorems and an auxiliary Lemma 7 are put in Appendix 3.

2. Minimal Weakly Stable Set, Uncovered Set and other concepts based on majority relation[1]

Majority relation and related concepts

A finite set A of alternatives is given, $|A|>2$. Throughout the paper lowercase letters (except CW) denote alternatives; capital letters denote sets of alternatives. Agents from a finite set $N=\{1, ..., n\}$, $|N|>1$ have preferences over alternatives from the set A. These preferences R_i ($i\in N$) are assumed to be complete binary relations on A. A relation R_i, $R_i\subseteq A\times A$ can be represented as a union of two relations P_i and I_i, $R_i=P_i\cup I_i$, $P_i\cap I_i=\varnothing$, one of which (P_i) is asymmetric ($\forall x, y\in A\ (x, y)\in P_i \Rightarrow (y, x)\notin P_i$), thus representing strong preference, the other one (I_i) is symmetric ($\forall x, y\in A\ xI_iy \Rightarrow yI_ix$), therefore it stands for a subrelation of indifference.

Majority relation is a binary relation μ, $\mu\subset A\times A$ constructed such that $(x, y)\in\mu$ if x is strongly preferred to y by majority, whichever defined, of all agents. For simple majority $x\mu y \Leftrightarrow \mathrm{card}\{i\in N\ xP_iy\}>\mathrm{card}\{i\in N\ yR_ix\}$. If xμy then it is said that x *dominates* y, and y *is dominated* by x. By assumption μ is asymmetric: $(x, y)\in\mu \Rightarrow (y, x)\notin\mu$.

If neither $(x, y)\in\mu$, nor $(y, x)\in\mu$ holds, then (x, y) is called a *tie*. A set of ties τ is a symmetric binary relation on A: $\tau\subseteq A\times A$, $(x, y)\in\tau \Rightarrow (y, x)\in\tau$. By definition $\mu\cap\tau=\varnothing$ and $\mu\cup\tau= A\times A$.

A relation μ is called a *tournament* if it is complete. Thus μ is a tournament when corresponding τ is empty, $\tau=\varnothing$. A tournament can be represented graphically by a complete asymmetric directed graph, where vertices correspond to alternatives, and directed lines (exactly one between each pair of vertices) represent majority relation, $x\to y \Leftrightarrow x\mu y$. By established convention a directed line is going from a dominating alternative to dominated one. In general case there are also ties. Ties are not connected by lines.

An ordered pair $x\to y$ is also called a *step*. A *path* $x\to y_1\to y_2\to\ldots\to y_{k-2}\to y_{k-1}\to y$ from x to y is an ordered sequence of steps starting at x and ending at y, such that the second alternative in each step coincides with the first alternative of the next step. In other words a path is an ordered sequence of alternatives $x, y_1, y_2, ..., y_{k-2}, y_{k-1}, y$, such that each alternative dominates the following one: $x\mu y_1, y_1\mu y_2, ..., y_{k-2}\mu y_{k-1}, y_{k-1}\mu y$. The number of steps in a path is called path's *length*. An alternative y is called *reachable in k steps* from x if there is a path of length k from x to y.

[1] The terminology, definitions and notation given in Section 2 are derived mainly from Aleskerov and Kurbanov (1999).

Lower contour set of an alternative x is a set L(x) of all alternatives dominated by x, $L(x)=\{y\in A: x\mu y\}$. Correspondingly, *upper contour set* of an alternative x is a set D(x) of all alternatives dominating x, $D(x)=\{y\in A: y\mu x\}$. A *horizon* of x is a set H(x) of all alternatives y, for which (x, y) is a tie, $H(x)=\{y\in A: x\tau y\}$. Obviously, $L(x)\cup D(x)\cup H(x)\cup\{x\}=A$.

Dominant, undominated and untrapped sets

A *Condorcet winner* CW, $CW\in A$, is an alternative dominating all other alternatives, $\forall x\in A,\ x\neq CW \Rightarrow CW\mu x$. *An alternative x is a weak Condorcet winner* iff it is not dominated by any other alternative, while there is at least one alternative that ties x, $\forall y\in A\ y\neq x \Rightarrow (x\mu y \text{ or } x\tau y)$ and $\exists z\in A\ z\neq x: x\tau z$.

A set of all undominated alternatives is called a *(majority) core* Cr, $x\in Cr \Leftrightarrow \forall y\in A\ y\neq x \Rightarrow (x\mu y \text{ or } x\tau y)$.

A set D, $D\subseteq A$, is called a *dominant set* (Ward, 1961; Smith, 1973) if each alternative in D dominates each alternative outside D, $\forall x,\ x\in D \Leftrightarrow \{\forall y\in A\backslash D \Rightarrow x\mu y\}$. A dominant set MD ($\equiv MD_{(1)}$) will be called a *minimal dominant set* (Schwartz, 1977) if none of its proper subsets is a dominant set. A set $MD_{(2)}$ is called a *minimal dominant set of the second degree* if it is a minimal dominant set for a set A\MD. $MD_{(i)}$ is a *minimal dominant set of the i'th degree* if it is a minimal dominant set for a set $A\backslash(\cup MD_{(j)})$, $1\leqslant j\leqslant i-1$.

A set U, $U\subseteq A$, is called an *undominated set* (Ward, 1961) if no alternative outside U dominates some alternative in U, $\forall x,\ x\in U \Leftrightarrow \{\forall y\in A\backslash D \Rightarrow (y, x)\notin\mu\}$. An undominated set MU is called a *minimal undominated set* (Schwartz, 1970) if none of its proper subsets is an undominated set. If such a set is not unique, then the social choice is defined as a union of these sets (Schwartz, 1972), which will likewise be denoted as MU. The union of minimal undominated sets (strong top cycle) is always a subset of minimal dominant set (weak top cycle), $MU\subseteq MD$. Evidently a core Cr is always a subset of MU, $Cr\subseteq MU$, since each {x}, {x}: $x\in Cr$, is a minimal undominated set.

It is said that x *traps* y if x dominates y and is not reachable from y via μ, xμy and there is no path from y to x (Duggan, 2007). An *untrapped set* (Duggan, 2007) UT is comprised of all alternatives that are not trapped. UT is always nonempty and is nested between the strong and weak top cycles, $MU\subseteq UT\subseteq MD$ (Duggan, 2007), consequently $Cr\subseteq UT$.

Lemma 1. If D_1 and D_2 are dominant sets then either $D_1\subseteq D_2$ or $D_2\subseteq D_1$.

Lemma 2. A minimal dominant set always exists and is unique.

The counterpart of Lemma 2 for tournaments was proved by Miller (1977).

It follows from the definitions that any dominant set is at the same time an undominated set. Thus Lemma 2 implies non-emptiness of MU, which implies non-emptiness of UT. For tournaments the notions of dominant and undominated sets coincide, i.e. MD=MU. Consequently, for tournaments the untrapped set coincides with the minimal dominant set (Duggan, 2007).

Lemma 3. If D is a dominant set then it is a direct sum of i minimal dominant sets of the first i degrees, $D=MD+MD_{(2)}+...+MD_{(j-1)}+MD_{(j)}+...+MD_{(i)}$.

According to Lemma 3 a set of all dominant sets in A for any μ might be represented as a sequence of s sets, $MD\subset D_{(2)}\subset...\subset D_{(i-1)}\subset D_{(i)}\subset...\subset D_{(s)}=A$,

$D_{(i)}=MD+MD_{(2)}+...+MD_{(j-1)}+MD_{(j)}+...+MD_{(i)}$, $D_{(i)}\setminus D_{(i-1)}=MD_{(i)}$.

By construction minimal dominant sets of different degree do not intersect, and the union of all these sets coincides with the set A. Consequently, any alternative belongs to one and only one set $MD_{(i)}$. Thus the hierarchy of dominant sets can be considered as a macrostructure of A, where all elements, i.e. alternatives, are distributed by "vertically" ordered layers $\{MD_{(i)}\}$.

Historical remark. Ward (1961) called dominant and undominated sets "majority sets" chosen under the "strong" and "weak" procedures respectively. Neither Ward (1961), no Smith (1973) formulated a condition of minimality for the sets they had introduced. The other names of the minimal dominant set are "minimal undominated set", "Condorcet set" (Miller, 1977); "GETCHA" (Schwartz, 1986); "weak top cycle". Though Fishburn (1977) does not define this concept explicitly, he speaks of "Smith's Condorcet principle", which is equivalent to choosing only alternatives from a minimal dominant set. The other names of the minimal undominated set are "GOCHA" (Schwartz, 1977, 1986); "undominated set" chosen under "Schwartz's rule" (Deb, 1977); "Schwartz choice set", "minimal externally undominated" set (Fishburn, 1977); "strong top cycle". If all individual preferences R_i are antisymmetric, $\forall i\ R_i=P_i$, and if majority is defined as consensus of agents, then the minimal dominant sets of different degrees $\{MD_{(i)}\}$ coincide with equivalence classes, defined by Kadane (1966).

Uncovered and uncaptured sets

Before a definition of an uncovered set is given, let us define the *covering* relation on A. It turns out that there are five different definitions of covering:

1) y covers x if yμx and $L(x)\subseteq L(y)\cup H(y)$ (Duggan, 2007), then x is uncovered $\Leftrightarrow \forall y$: yμx $\Rightarrow \exists z$: xμz & zμy;

2) y covers x if yμx and $L(x)\subseteq L(y)$ (Miller, 1980), then x is uncovered $\Leftrightarrow$ $\forall y$: yμx $\Rightarrow \exists z$: (xμz&zμy) or (xμz&zτy));

3) y covers x if yμx and $D(y)\subseteq D(x)$ (Fishburn, 1977; Miller, 1980), then x is uncovered $\Leftrightarrow \forall y$: yμx $\Rightarrow \exists z$: (xμz & zμy) or (xτz & zμy);

4) y covers x if yμx and $L(x)\subseteq L(y)$ & $D(y)\subseteq D(x)$ (Miller, 1980; McKelvey, 1986), then x is uncovered $\Leftrightarrow \forall y$: yμx $\Rightarrow \exists z$: (xμz & zμy) or (xμz &zτy) or (xτz & zμy);

5) y covers x if yμx and $H(x)\cup L(x)\subseteq L(y)$ (Duggan, 2007), then x is uncovered $\Leftrightarrow \forall y$: yμx $\Rightarrow \exists z$: (xμz & zμy) or (xμz & zτy) or (xτz & zμy) or (xτz & zτy).

The definitions are listed according to their relative "strength": the strength of covering decreases, and the number of uncovered alternatives correspondingly increases with increase of the definition's number. It follows from the definitions of the upper and lower contour sets and from transitivity of the relation of inclusion that the relation of covering under the second, third, fourth and fifth definitions is transitive. Cycles of covering are possible under the first definition. For a tournament all five definitions of covering are equivalent. Relation of covering has no symmetric component under all five definitions listed above, i.e. if x covers y then it is not possible for y to cover x.

The *uncovered set* (Fishburn, 1977; Miller, 1980) UC is comprised of all alternatives that are not covered. An uncovered set, whichever defined, is unique. Let UC^{I}, UC^{II}, UC^{III}, UC^{IV} and UC^{V} denote uncovered sets under the first, second, third, fourth and fifth definitions of covering respectively. Evidently, $UC^{IV}=UC^{II}\cup UC^{III}$, $UC^{I}\subseteq UC^{II}\subseteq UC^{IV}$ $UC^{I}\subseteq UC^{III}\subseteq UC^{IV}$ and $UC^{IV}\subseteq UC^{V}$. UC^{II} and UC^{III} are not logically nested. Let us consider the following example: A={a, b, c, x, y, z}, μ={(a, b), (b, c), (c, a), (x, a), (x, b), (x, y), (y, z), (z, b), (z, x)} (see Figure 1). Here UC^{II}={b, c, x, y, z}, UC^{III}={a, c, x, y, z}, $UC^{II}\setminus UC^{III}$={b}; $UC^{III}\setminus UC^{II}$={a}.

If a relation is transitive, it always possesses maximal elements. Therefore sets UC^{II}, UC^{III}, UC^{IV} and UC^{V} are always non-empty, while UC^{I} may be empty. For instance, let A={a, b, c, d}, μ={(a, b), (b, c), (c, d), (d, a)}. According to the first definition of covering a covers b, b covers c, c covers d and d covers a, i.e. there is a cycle of covering, including all alternatives in A. Therefore $UC^{I}=\varnothing$.

Since any undominated alternative is by definition uncovered, a core is a subset of any UC, $Cr\subseteq UC$. It is possible that the inclusion is strict, $Cr\subset UC$. For instance, let A={a, b, c, d}, μ={(a, b), (b, c), (c, d), (d, b)} then Cr={a}, UC^{I}={a, c, d}.

The relation $UC\subseteq MD$ holds for all five versions of covering, whereas MU and all UC are not logically nested in general case: in tournaments $UC\subseteq MU=MD$, while digraph on Figure 1 shows that $MU\subset UC^{I}$ is also possible: MU={x, y, z}; UC^{I}={c, x, y, z}, $UC^{IV}=UC^{V}$=A. This example also proves that the sets UC^{II}, UC^{III}, UC^{IV} and UC^{V} are not logically nested with UT: in tournaments

$UC^{II}=UC^{III}=UC^{IV}=UC^{V}\subseteq UT=MD$, while $UT\subset UC^{II}$, $UT\subset UC^{III}$ $UT\subset UC^{IV}$ and $UT\subset UC^{V}$ are also possible, since here $UT=MU=\{x, y, z\}$. By the first definition of covering UC^{I} is always a subset of UT, $UC^{I}\subseteq UT$.

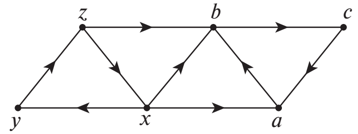

Fig. 1

An alternative x *captures* an alternative y, if x 1) covers y according to the fourth definition of covering and 2) covers all alternatives from the lower contour set of y according to the third definition of covering, $D(x)\subseteq D(z)$, $\forall z\in L(y)$ (Duggan, 2007). Then x is uncaptured $\Leftrightarrow$ $\forall y$: $y\mu x \Rightarrow$

either $\exists z$: $(x\mu z\ \&\ z\mu y)$ or $(x\mu z\ \& z\tau y)$ or $(x\tau z\ \&\ z\mu y)$;

or $\exists v, w$: $(x\mu v\ \&\ v\mu w\ \&\ w\mu y)$ or $(x\mu v\ \&\ v\tau w\ \&\ w\mu y)$.

An *uncaptured set* (Duggan, 2007) UCp is comprised of all alternatives that are not captured. By definition $UC^{IV}\subseteq UCp$, consequently, $Cr\subseteq UCp$, $UC^{I}\subseteq UCp$, $UC^{II}\subseteq UCp$, $UC^{III}\subseteq UCp$ and UCp is always nonempty. The uncaptured set is a subset of the minimal dominant set, $UCp\subseteq MD$ (Duggan, 2007). UCp is not logically nested with UC^{V}, UT and MU: in tournaments $UC\subseteq UCp\subseteq MU=UT=MD$, while $UCp\subset UC^{V}$, $MU\subset UCp$ and $UT\subset UCp$ are also possible. The possibility of inclusions $MU\subset UCp$ and $UT\subset UCp$ is again proved by the graph from Figure 1, since there UCp=A. To show the possibility of $UCp\subset UC^{V}$ let us consider another example: $A=\{a, b, c, x, y, z\}$, $\mu=\{(a, b), (a, c), (b, c), (x, y), (x, z), (y, z), (z, c)\}$ (see Figure 2). Here $UCp=\{c, x, z\}$ and $UC^{V}=A$.

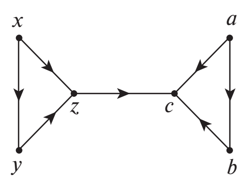

Fig. 2

Historical remark. In fact, both Fishburn and Miller did not include a demand for $y\mu x$ into their definitions. Respectively Miller (1980, p. 94) proposes only $L(x)\subseteq L(y)\ \&\ D(y)\subseteq D(x)$ as a definition of covering for general case, $\tau\neq\varnothing$. For tournaments this discrepancy does not make any difference, but in general case it does. The condition $y\mu x$ in the definitions of covering is what

makes relation of covering asymmetric. If it is dropped then one gets five more versions of covering and uncovered sets. These "new" relations may possess a symmetric component. For instance, if $x\tau y$ and $L(x)=L(y)$ then under original Miller's definition of covering (without demand for $x\mu y$ or $y\mu x$) x cover y and y covers x. The condition $y\mu x$ in the definitions of covering is also needed for them to be consistent with the concept of stability (especially when stability is interpreted dynamically) and with a definition of the uncaptured set.

The term "uncovered set" was introduced by Miller. Fishburn talks of an "image of Fishburn's social choice function".

Weakly stable sets

A set WS is called a *weakly stable set* (Aleskerov, Kurbanov, 1999) if it has the following property: if x belongs to a weakly stable set, then for any alternative y outside the weakly stable set, which dominates x, there is an alternative z in the weakly stable set, which dominates y, $\forall x\in A$, $x\in WS \Leftrightarrow (\exists y\notin WS$: $y\mu x \Rightarrow \exists z\in WS$: $z\mu y)$. In terms of D(x) and L(x) WS is weakly stable $\Leftrightarrow$ $(\forall y\notin WS$ $WS\cap L(y)\neq\varnothing \Rightarrow WS\cap D(y)\neq\varnothing)$. A weakly stable set MWS is called a *minimal weakly stable set* if none of its proper subsets is a weakly stable set. If such set is not unique, then the social choice is defined as a union of these sets (Aleskerov, Kurbanov, 1999), which will likewise be denoted MWS.

According to the definition if x is undominated then $\{x\}$ is a minimal weakly stable set, therefore a core is a subset of MWS, $Cr\subseteq MWS$. The inclusion may be strict, $Cr\subset MWS$. For instance, let $A=\{a, b, c, d\}$, $\mu=\{(a, b), (b, c), (c, d), (d, b)\}$ then $Cr=\{a\}$, $MWS=\{a, c, d\}$. It also follows from the definitions that any dominant set is at the same time a weakly stable set. Thus Lemma 2 implies non-emptiness of MWS.

Lemma 4. $MWS\subseteq MD$.

Corollary. Since MU, MWS, all UC (except UC^{I}), UCp, UT are always nonempty, then if there is a Condorcet winner CW, all sets coincide with a core, which contains only one alternative — CW, $MD=MU=MWS=UC=UCp=UT=Cr=\{CW\}$. It follows from the definition of UC^{I} that the same also holds for UC^{I}.

The definition of a minimal weakly stable set proposed by Aleskerov and Kurbanov is global. For practical calculations one needs a criterion to determine whether an alternative belongs to a minimal weakly stable set or not. For tournaments such criterion is given by Theorem 1. But before that, two important properties of weakly stable sets should be established.

Lemma 5. If μ is a tournament, then B is a weakly stable set iff $\forall y\notin B \Rightarrow B\cap D(y)\neq\varnothing$. That is B is a weakly stable set iff there is one-step path from some alternative in B to any alternative outside B.

Corollary (monotonicity). Let B⊆C. If B is a weakly stable set then C is a weakly stable set. If C is not a weakly stable set then B is not a weakly stable set.

Theorem 1. If μ is a tournament, then an alternative x belongs to a union of minimal weakly stable sets MWS iff 1) either x is uncovered or 2) some alternative from x's lower contour set L(x) is uncovered.

Corollary. For tournaments, the uncovered set is a subset of the union of minimal weakly stable sets, UC⊆MWS⊆MD⊆A.

It is worth noting that there are tournaments for which inclusion is strict, UC⊂MWS⊂MD⊂A. For example let A={a, b, c, d, e, f} and μ={(a, c), (a, d), (a, e), (a, f), (b, a), (b, d), (b, e), (b, f), (c, b), (c, e), (c, f), (d, c), (d, f), (e, d), (e, f)} (see Figure 3), then UC={a, b, c}, MWS={a, b, c, d} and MD={a, b, c, d, e}.

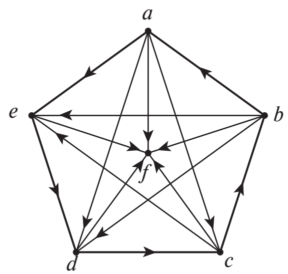

Fig. 3

Corollary of Lemma 5 and the proof of Theorem 1 depend on the assumption that μ is a tournament through the proposition of Lemma 5 only. That is in general case if Lemma 5 holds then so do Corollary and Theorem 1. Thus it is possible to take Lemma 5 as a *second definition of a weakly stable set*: a set WS is weakly stable if ∀y∉WS ⇒ WS∩D(y)≠∅. Let MWS^{I} and MWS^{II} denote a union of minimal weakly stable set under the old and new definitions of weak stability correspondingly. In a tournament these definitions of weak stability are equivalent.

Theorem 1 shall be restated as

Theorem 1a. For any majority relation μ an alternative x belongs to a minimal stable set MWS^{II} iff x is uncovered according to the third definition of covering or some alternative from the lower contour of x is uncovered according to the third definition of covering, $x \in MWS^{II} \Leftrightarrow x \in UC^{III}$ or ∃y: $y \in L(x)$ & $y \in UC^{III}$.

Corollary. A union of minimal weakly stable sets MWS^{II} is a superset for an uncovered set UC^{III} (and thus for a core) and a subset of an uncaptured set UCp, $Cr \subseteq UC^{III} \subseteq MWS^{II} \subseteq UCp$.

There are μ such that $MWS^{II} \subset UCp$, for example one depicted on Figure 2. There MWS^{II}={a, x} amd UCp={a, x, z}.

If μ is not assumed to be a tournament, then corresponding modification in the proof of Lemma 5 (a change of D(x)∪L(x)∪{x}=A for D(x)∪L(x)∪H(x)∪{x}=A) yields the following consideration. Aleskerov-Kurbanov's definition of a weakly stable set (x∈WS ⇔ (∃y∉WS: yμx ⇒ ∃z∈WS: zμy)) implies that B is weakly stable iff ∀y∉B B∩D(y)≠∅ or B⊆H(y). That is a set, which is weakly stable by the second definition, is weakly stable according to the original version of weak stability. At the same time there are sets not weakly stable by the former definition but weakly stable according to the latter one, namely, when ∀y∉B ⇒ (B∩D(y)≠∅ or B∈H(y)) and ∃z∉B: B∈H(z). Moreover, in general case weak stability retains its monotonicity only under the second definition, whereas under the original version it is possible for some B, C, D: D⊂C⊂B⊂A that C may not be a weakly stable set while B and D are weakly stable. As a result, a weakly set B, which is minimal according to the second definition may not be minimal according to the first one, since it may contain a proper subset C, C⊂B, such that it is weakly stable by the first definition and is not weakly stable by the second one.

Lemma 6 establishes logical relations for the rest of all pairs of sets introduced so far.

Lemma 6. MWS^{I} is not logically nested with UC^{I}, UC^{II}, UC^{III}, UC^{IV}, UC^{V}, MWS^{II}, MU; $MWS^{I} \subseteq UCp$ and ∃μ: $MWS^{I} \subset UCp$; $MWS^{I} \subseteq UT$ and ∃μ: $MWS^{I} \subset UT$; MWS^{II} is not logically nested with UC^{II}, UC^{IV}, UC^{V}, MU and UT; $UC^{I} \subseteq MWS^{II}$; $MWS^{II} \subseteq MD$.

Table 2 and Table 3 in Conclusion summarize the relations between all twelve sets Cr, UC^{I}, UC^{II}, UC^{III}, UC^{IV}, UC^{V}, MWS^{I}, MWS^{II}, UCp, MU, UT and MD for general case and for tournaments respectively.

3. k-stable[2] alternatives, k-stable sets and their relationship

From now on, unless it is specifically noted, only tournaments will be considered.

k-stable points

We can deepen our understanding of the uncovered and weakly stable sets and generalize these notions if we consider the relative stability of alternatives

[2] The notion and the general idea were proposed by F. Aleskerov.

and sets of alternatives. An alternative x will be called *generally stable* if every other alternative in A is reachable from x, otherwise x is *unstable*.

Remark. Thus defined stability is also open for a dynamic interpretation. It is natural to view a certain position (a state of a system) as more stable one if it is less subject to change. In a voting game a "state of a system" is either an alternative, which is a status quo, or a certain set of alternatives, to which a status quo belongs. Thus a state is less subject to change if it takes less effort (steps, rounds of voting) to return to the same state (to adopt the same alternative or an alternative from the same set of alternatives by the process of consequent voting) after a "perturbation", i.e. after a status quo was outvoted and changed.

Every alternative in A is reachable from x iff x belongs to a minimal dominant set (Miller, 1977), thus all alternatives of a minimal dominant set and only they are generally stable.

Since A is finite, if y is reachable from x, then there is a path from x to y with a minimal length. Let l(x, y) denote a *minimal length function*. The function l(x, y) has the following property: $l(x, y)>1 \Rightarrow l(y, x)=1$.

For x and y, such that $x\in D$, $y\in A\backslash D$, where D is a dominant set, l(y, x) is not defined, as x is not reachable from y. For such cases let $l(y, x)=\infty$. If x belongs to a minimal dominant set, l(x, y) is defined and has a finite value for all $y\in A\backslash\{x\}$. Let $l(x, x)=0$ for l(x, y) to be defined on the whole set A. In terms of l(x, y) x is generally stable when $\forall y\in A\ l(x, y)<\infty$.

Let $l_{max}(x)$ denote a function of x defined as $l_{max}(x)=\max_{y\in A} l(x, y)$. If $l_{max}(x)=k<\infty$ then it is possible to reach any alternative in A from x in no more than k steps, but there is at list one alternative reachable from x in less than k steps. The function $l_{max}(x)$ may serve as a measure of stability and thus helps in comparison of alternatives by their stability and in separation of them by classes of stability. Therefore, let the value of $l_{max}(x)$ be called a *degree of stability* of x. If the degree of stability of an alternative x is k, $k<\infty$, x will be called *k-stable*. Let $SP_{(k)}$ denote a class of k-stable points, i.e. a set of all k-stable alternatives in A, $x\in SP_{(k)} \Leftrightarrow l_{max}(x)=k$. An algorithm for calculating the classes of k-stable alternatives and the minimal dominant sets is given in Appendix 2.

An alternative x has the degree of stability k=1 iff x is a Condorcet winner, x=CW. Therefore $SP_{(1)}=\{CW\}$. It is also evident that if $SP_{(1)}\neq\varnothing$, then all $SP_{(k>1)}=\varnothing$, since CW is not reachable from any other alternative.

An alternative x has the degree of stability k=2 iff x is an uncovered alternative, i.e. $SP_{(2)}$ is an uncovered set UC, $SP_{(2)}=UC$ (Miller, 1980).

By construction the classes of stable alternatives do not intersect, $SP_{(i)}\cap SP_{(j)}=\varnothing$, $i\neq j$. Since all alternatives that are generally stable belong to a minimal dominant set MD, and all alternatives from MD are generally stable, MD is a direct sum of all classes of k-stable alternatives, $MD=SP_{(1)}+SP_{(2)}+SP_{(3)}+\ldots+SP_{(k)}+\ldots$ Since A is finite, there is a generally stable alternative (at least one), the degree of stability of which is maximal $m=\max_{x\in MD} l_{max}(x)$. It follows immediately that 1) $\forall k>m\ SP_{(k)}=\varnothing$; 2) $SP_{(m)}\neq\varnothing$; 3) $MD=SP_{(1)}+SP_{(2)}+SP_{(3)}+\ldots+SP_{(m)}$.

Theorem 2. (Nonemptiness of point-classes) If there is no Condorcet winner, each class of k-stable alternatives with the degree k equal or less than maximal is nonempty, except $SP_{(1)}$, $\forall SP_{(k)}\neq\varnothing$, $2\leqslant k\leqslant m=\max_{x\in MD} l_{max}(x)$.

Finally, let $P_{(k)}$ denote a set of those generally stable alternatives, from which it is possible to reach any given alternative in A in *no more* than k steps. By definition $P_{(k)}=SP_{(1)}+SP_{(2)}+\ldots+SP_{(k)}$. According to the definition of the capturing relation, x is uncaptured if it is possible to reach any other alternative in A in no more than 3 steps, thus $P_{(3)}$ is the uncaptured set UCp, $P_{(3)}=SP_{(1)}+SP_{(2)}+SP_{(3)}=UCp$

Therefore the following system of subsets can be defined in a minimal dominant set.

$P_{(1)}=\{CW\}=MD$; if $P_{(1)}=\varnothing$, then

$P_{(2)}=UC\neq\varnothing$, an uncovered set;

$P_{(3)}=UCp$, an uncaptured set;

$P_{(1)}\subset P_{(2)}\subset P_{(3)}\subset\ldots\subset P_{(m-1)}\subset P_{(m)}=MD$, $m=\max_{x\in MD} l_{max}(x)$, all inclusions are strict according to Theorem 2.

k-stable sets

Similarly to alternatives, a set of alternatives X will be called *generally stable set* if it is possible to reach any alternative outside X from some alternative in X, otherwise X is *unstable*. An alternative y outside X is *reachable in k steps from X* if there is a k-step path to y from some x in X. Since all alternatives are reachable from alternatives in a minimal dominant set, but alternatives in MD are not reachable from outside, any set X, which has nonempty intersection with MD, $\forall X: X\cap MD\neq\varnothing$, is generally stable, otherwise it is unstable.

In terms of l(x, y), X is generally stable if $\forall y\in A\backslash X\ \exists x\in X: l(x, y)<\infty$. A function $l(X, y)=\min_{x\in X} l(x, y)$ for any $y\in A\backslash X$ will be called a *minimal length function*, which represents a distance from a given set X to a given alternative y outside X. $l(X, y)=\infty$ when y is not reachable from X. Let $l(X, y)=0$ if $y\in X$.

Correspondingly, $l_{max}(X)=\max_{y\in A} l(X, y)$. If $l_{max}(X)=k<\infty$ then $\forall y\in A\ \exists x\in X: l(x, y)\leqslant k\ \&\ \exists y\in A\backslash X: \forall x\in X\ l(x, y)\geqslant k$.

The value of $l_{max}(X)$ will be called the *degree of stability* of a set of alternatives X. If the degree of stability of X is k, the set X will be called *k-stable*. A k-stable set will be called a *minimal k-stable set* if none of its proper subsets is a k-stable set.

Let $SS_{(k)}$ denote a class of those alternatives, which belong to some minimal k-stable set, but do not belong to any minimal stable set with the degree of stability less than k. By construction these classes do not intersect, $\forall i \neq j$ $SS_{(i)} \cap SS_{(j)} = \varnothing$.

Finally, let $S_{(k)}$ denote a union of those minimal generally stable sets, from which it is possible to reach any alternative outside a set in *no more* than k steps.

Evidently, $S_{(k)}=SS_{(1)}+SS_{(2)}+...+SS_{(k)}$

It follows from Lemma 5 that a k-stable set of degree k=1 is equivalent to a weakly stable set. Therefore $S_{(1)}$ coincides with the union of minimal weakly stable sets MWS, $S_{(1)}$=MWS.

There is a relationship between all these sets, which were introduced above on a basis of an idea of stability, the relationship similar to one, which is established for the uncovered set and the union of the weakly stable sets by Theorem 1. Theorem 3 determines this relationship.

Theorem 3. 1) $P_{(2)} \subseteq S_{(1)} \subseteq P_{(3)}$ (i.e. UC⊆MWS⊆UCp); $\exists \mu$: $P_{(2)} \subset S_{(1)} \subset P_{(3)}$;
2) $\forall k$: k>1 $P_{(k)} \subseteq S_{(k)} \subseteq P_{(k+2)}$.

Corollary 1. $\forall k$: k>3 $x \in SS_{(1)} \Rightarrow (x \in SP_{(2)}$ or $x \in SP_{(3)})$ & $x \notin SP_{(k)}$.

Corollary 2. $\forall k$: k=2 or k>4 $x \in SS_{(2)} \Rightarrow (x \in SP_{(3)}$ or $x \in SP_{(4)})$ & $x \notin SP_{(k)}$.

Corollary 3. $\forall i$: i>k+2 or i<k $x \in SS_{(k)}$, $\forall k$: k>2 $\Rightarrow (x \in SP_{(k)}$ or $x \in SP_{(k+1)}$ or $x \in SP_{(k+2)})$ & $x \notin SP_{(i)}$.

Corollary 4. $\forall i$: i>k or i<k-2 $x \in SP_{(2)} \Rightarrow x \in SS_{(1)}$; $x \in SP_{(k)}$, $\forall k$: k>2 $\Rightarrow (x \in SS_{(k-2)}$ or $x \in SS_{(k-1)}$ or $x \in SS_{(k)})$ & $x \notin SS_{(i)}$.

Finally, let us consider the following examples: A={a, b, v, w, x, y, z};

μ_1={(a, b), (a, w), (a, x), (a, y), (a, z), (b, v), (b, x), (b, y), (b, z), (v, a), (v, w), (v, x), (v, z), (w, b), (w, x), (w, y), (x, y), (x, z), (y, v), (y, z), (z, w)} (Figure 4);

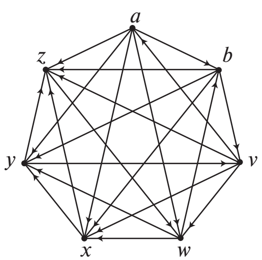

Fig. 4

μ_2={(a, b), (a, w), (a, x), (a, y), (a, z), (b, v), (b, x), (b, y), (b, z), (v, a), (v, w), (v, x), (v, z), (w, b), (w, x), (w, y), (x, y), (x, z), (y, v), (z, w), (z, y)} (Figure 5);

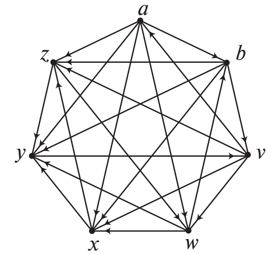

Fig. 5

μ_3={(a, b), (a, w), (a, x), (a, y), (a, z), (b, v), (b, x), (b, y), (b, z), (v, a), (v, x), (v, z), (w, b), (w, v), (w, x), (w, y), (x, y), (x, z), (y, v), (z, w), (z, y)} (Figure 6).

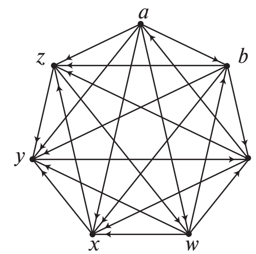

Fig. 6

For each digraph Table 1 shows distribution of alternatives by point-classes $\{SP_{(k)}\}$ and set-classes $\{SS_{(k)}\}$.

*Table 1. Dis*tribution of points by point-classes $\{SP_{(k)}\}$ and set-classes $\{SS_{(k)}\}$

Figure 4			**Figure 5**				**Figure 6**		
	$SP_{(2)}$	**$SP_{(3)}$**		**$SP_{(2)}$**	**$SP_{(3)}$**	**$SP_{(4)}$**		**$SP_{(2)}$**	**$SP_{(3)}$**
$SS_{(1)}$	a, b, v	y, w	**$SS_{(1)}$**	a, b, v	y, w		**$SS_{(1)}$**	a, b, v, w	y, z
$SS_{(2)}$		z	**$SS_{(2)}$**			z	**$SS_{(2)}$**		
$SS_{(3)}$		x	**$SS_{(3)}$**		x		**$SS_{(3)}$**		x

These examples show that all three theoretically possible options hold: $x \in (SS_{(k-2)}, SP_{(k)})$, $x \in (SS_{(k-1)}, SP_{(k)})$, $x \in (SS_{(k)}, SP_{(k)})$. Therefore the statement of Theorem 3 can not be made stronger. Evidently, the inclusion $P_{(2)} \subseteq S_{(1)}$ is just a "boundary effect", and it can not be generalized.

Together with Example 3 they also prove that the cases when $SS_{(k)}=\varnothing$, or even $SS_{(k-1)}=\varnothing$ & $SS_{(k)}=\varnothing$ are possible. Therefore there is no counterpart of Theorem 2 for set-classes $\{SS_{(k)}\}$. The cases where $SS_{(k-2)}=\varnothing$, $SS_{(k-1)}=\varnothing$, $SS_{(k)}=\varnothing$ for any k: $k \leqslant m$, $m=\max_{x \in MD} l_{max}(x)$ are impossible, since $SS_{(k-2)}=\varnothing$, $SS_{(k-1)}=\varnothing$, $SS_{(k)}=\varnothing$ implies $SP_{(k)}=\varnothing$, a contradiction with Theorem 2.

It is important to note that $P_{(k)} \subseteq S_{(k)} \subseteq P_{(k+2)}$ does not imply that either $P_{(k)} \subseteq S_{(k)} \subseteq P_{(k+1)}$ or $P_{(k+1)} \subseteq S_{(k)} \subseteq P_{(k+2)}$ holds. The distribution table of a digraph from Figure 5 demonstrates that there are cases with such pairs of alternatives (x, y), where one alternative belongs to a point-class of greater degree and to a set-class of lesser degree than the other one, $x \in (SS_{(k-2)}, SP_{(k)})$, $y \in (SS_{(k-1)}, SP_{(k-1)})$. Therefore, even though it is possible to compare all alternatives by stability using point-classes $\{SP_{(k)}\}$ or set-classes $\{SS_{(k)}\}$ independently, there is no natural aggregated order: one may call x more stable than y when $x \in (SS_{(k)}, SP_{(l)})$, $y \in (SS_{(m)}, SP_{(n)})$, $k \leqslant m$, $l \leqslant n$ & ($k<m$ or $l<n$), but it is impossible to compare alternatives $x \in (SS_{(k-2)}, SP_{(k)})$ and $y \in (SS_{(k-1)}, SP_{(k-1)})$.

4. Conclusion

In the rational choice paradigm the main problem is a general absence of a core. The core exists so rarely, that one needs 1) either to make quite restrictive assumptions with regard to a space of individual preferences to guarantee its existence, 2) or to find a solution concept, which can be used as a substitute. For instance, in the spatial theory of voting a notion of ideology is used as a means to make an issue space one-dimensional (Ferejohn, 1995) in order to ensure the existence of a median voter, who's ideal point is a Condorcet winner. In the multi-dimensional setting the median voter exists only under non-robust Plott's pairwise symmetry conditions (Plott, 1967) for majority rule equilibrium. In this paper the latter approach was chosen.

Several such solutions concepts were considered and compared: Cr, UC^{I}, UC^{II}, UC^{III}, UC^{IV}, UC^{V}, MWS^{I}, MWS^{II}, UCp, MU, UT and MD. Tables 2 and 3 summarize their relations for general case and for tournaments respectively. The symbol "⊆" in a cell points out that a set of a corresponding row R is always a subset of a set of a corresponding column C, $R \subseteq C$. The symbol "n.n." points out that sets R and C are not logically nested. The symbol "=" points out that sets R and C are equivalent.

Table 2. General case

	UC^{I}	UC^{II}	UC^{III}	UC^{IV}	UC^{V}	MWS^{I}	MWS^{II}	UCp	MU	UT	MD
Cr	⊆	⊆	⊆	⊆	⊆	⊆	⊆	⊆	⊆	⊆	⊆
UC^{I}		⊆	⊆	⊆	⊆	n.n.	⊆	⊆	n.n.	⊆	⊆
UC^{II}			n.n.	⊆	⊆	n.n.	n.n.	⊆	n.n.	n.n.	⊆
UC^{III}				⊆	⊆	n.n.	⊆	⊆	n.n.	n.n.	⊆
UC^{IV}					⊆	n.n.	n.n.	⊆	n.n.	n.n.	⊆
UC^{V}						n.n.	n.n.	n.n.	n.n.	n.n.	⊆
MWS^{I}							n.n.	⊆	n.n.	⊆	⊆
MWS^{II}								⊆	n.n.	n.n.	⊆
UCp									n.n.	n.n.	⊆
MU										⊆	⊆
UT											⊆

Table 3. Tournaments

	UC^{I}	UC^{II}	UC^{III}	UC^{IV}	UC^{V}	MWS^{I}	MWS^{II}	UCp	MU	UT	MD
Cr	⊆	⊆	⊆	⊆	⊆	⊆	⊆	⊆	⊆	⊆	⊆
UC^{I}		=	=	=	=	⊆	⊆	⊆	⊆	⊆	⊆
UC^{II}			=	=	=	⊆	⊆	⊆	⊆	⊆	⊆
UC^{III}				=	=	⊆	⊆	⊆	⊆	⊆	⊆
UC^{IV}					=	⊆	⊆	⊆	⊆	⊆	⊆
UC^{V}						⊆	⊆	⊆	⊆	⊆	⊆
MWS^{I}							=	⊆	⊆	⊆	⊆
MWS^{II}								⊆	⊆	⊆	⊆
UCp									⊆	⊆	⊆
MU										=	=
UT											=

Miller (1977) proved for tournaments that outcomes for some important majority voting games are localized in the minimal dominant set of alternatives. Here it was demonstrated that any set of alternatives A with a majority relation μ defined over it possesses an internal structure created by a hierarchy of minimal dominant sets $\{MD_{(i)}\}$, which at the same time is a system of all dominant sets $\{D_{(i)}\}$ in A with given μ.

$D_{(1)} \subset D_{(2)} \subset \ldots \subset D_{(i-1)} \subset D_{(i)} \subset \ldots \subset D_{(s)} = A$,

$D_{(i)} = MD + MD_{(2)} + \ldots + MD_{(j-1)} + MD_{(j)} + \ldots + MD_{(i)}$, $D_{(i)} \setminus D_{(i-1)} = MD_{(i)}$.

The highest level of this hierarchy is the minimal dominant set proper MD.

Miller (1980) introduced the concept of an uncovered set. Aleskerov and Kurbanov (1999) defined the minimal weakly stable set. Here it was demonstrated that the minimal weakly stable set is related to the notion of covering and to the uncovered set. It was shown that these two concepts are intimately connected with an idea of stability.

As an attempt to generalize the notions of the uncovered set and the minimal weakly stable set the concepts of k-stable alternatives and k-stable sets of alternatives were introduced. It was demonstrated that the systems of classes of k-stable alternatives $\{SP_{(k)}\}$ (point-classes) and classes of minimal k-stable sets $\{SS_{(k)}\}$ (set-classes) form substructures in the minimal dominant set similar to the structure that dominant sets create in the universal set A, i. e.

1) $P_{(2)} \subset P_{(3)} \subset \ldots \subset P_{(m-1)} \subset P_{(m)} = MD$, $P_{(k)} = SP_{(1)} + SP_{(2)} + \ldots + SP_{(k)}$;
2) $S_{(1)} \subset S_{(2)} \subset S_{(3)} \subset \ldots \subset S_{(n-1)} \subset S_{(n)} = MD$, $S_{(k)} = SS_{(1)} + SS_{(2)} + \ldots + SS_{(k)}$.

The uncovered set and the union of minimal weakly stable sets are the highest levels in the hierarchies of point-classes and set-classes, $P_{(2)}$ and $S_{(1)}$, respectively. It also turned out that the second element in the first hierarchy is the uncaptured set: $P_{(3)} = UCp$. Corollary of Theorem 1 was generalized and it was demonstrated that these systems of classes are related to each other through the relation of inclusion $P_{(2)} \subseteq S_{(1)} \subseteq P_{(3)}$ and $P_{(k)} \subseteq S_{(k)} \subseteq P_{(k+2)}$ for any k: k>1. It was also found that all point-classes $\{SP_{(k)}\}$ for any k: $2 \leqslant k \leqslant k_{max}$, are always nonempty for tournaments, which is not true about the set-classes $\{SS_{(k)}\}$.

Although alternatives and sets not included in the minimal dominant set were defined as unstable, their "instability" is being of different degree. Since $MD_{(i)}$ is the minimal dominant set in $A \setminus D_{(i-1)}$, one can measure the difference in stability of all points in A, not only those in MD, by defining similar systems of point-classes and set-classes for all $MD_{(i)}$, not only for the minimal dominant set. As a result the system of dominant sets and systems of point-classes and set-classes represent respectively macro-scale structure and micro-scale substructure of a universal set. Since the classes do not intersect, and their hierarchies cover the whole set A, for any tournament each alternative will be characterized by three numbers k, l, m, as belonging 1) to a minimal dominant set of k'th degree $MD_{(k)}$, 2) to a class of l-stable points $SP_{(l)}$ and 3) to a class of minimal m-stable sets $SS_{(m)}$ in $MD_{(k)}$. That is for tournaments, the hierarchy of dominant sets and respective hierarchies of classes of k-stable points and classes of minimal k-stable sets create a system of reference based on the principle of stability. Therefore one may assess the relative stability of alternatives by comparing their coordinates in this system.

Correspondingly each tournament (each complete digraph) is characterized by a distribution table, i.e. a table of distribution of alternatives (points) by classes, similar to Tables 1 and 4.[3]

Appendix 1

Example 1. (See Figure 1 in Section 2) $\exists\mu$:
$MWS^{I} = MU \subset UC^{I} = UT \subset \{UC^{II}; UC^{III}\} \subset UC^{IV} = UC^{V} = MWS^{II} = UCp = MD$; $UC^{II} \setminus UC^{III} \neq \varnothing$ & $UC^{III} \setminus UC^{II} \neq \varnothing$.
A={a, b, c, x, y, z}
μ={(a, b), (b, c), (c, a), (x, a), (x, b), (x, y), (y, z), (z, b), (z, x)}
$MWS^{I} = MU = \{x, y, z\}$; $UC^{I} = UT = \{c, x, y, z\}$; $UC^{II} = \{b, c, x, y, z\}$; $UC^{III} = \{a, c, x, y, z\}$; $UC^{II} \setminus UC^{III} = \{b\}$; $UC^{III} \setminus UC^{II} = \{a\}$; $UC^{IV} = UC^{V} = MWS^{II} = UCp = MD = A$

Example 2. (See Figure 2 in Section 2) $\exists\mu$:
$UC^{I} = UC^{III} = MWS^{I} = MWS^{II} = MU = UT \subset UC^{II} = UC^{IV} = UCp \subset UC^{V} = MD$
A={a, b, c, x, y, z}
μ={(a, b), (a, c), (b, c), (x, y), (x, z), (y, z), (z, c)}
$UC^{I} = UC^{III} = MWS^{I} = MWS^{II} = MU = UT = \{a, x\}$; $UC^{II} = UC^{IV} = UCp = \{a, x, z\}$; $UC^{V} = MD = A$

Example 3. (See Figure 3 in Section 2)
$\exists\mu$: $UC \subset MWS \subset MD \subset A$
A={a, b, c, d, e, f}
μ={(a, c), (a, d), (a, e), (a, f), (b, a), (b, d), (b, e), (b, f), (c, b), (c, e), (c, f), (d, c), (d, f), (e, d), (e, f)}
UC={a, b, c}, MWS={a, b, c, d}, MD={a, b, c, d, e}.

Table 4. Distribution of points by point-classes $\{SP_{(k)}\}$ and set-classes $\{SS_{(k)}\}$

	$SP_{(2)}$	$SP_{(3)}$	$SP_{(4)}$
$SS_{(1)}$	a, b, c	d	
$SS_{(2)}$			
$SS_{(3)}$			
$SS_{(4)}$			e

[3] Since all relevant definitions were based on the idea of proximity, the numbers of points in classes $\{MD_{(k)}\}$, $\{SP_{(k)}\}$ and $\{SS_{(k)}\}$ are graph invariants.

Example 4. (see Figure 4)

$\exists\mu$: $P_{(2)} \subset S_{(1)} \subset P_{(3)}$

A={a, b, v, w, x, y, z}

μ={(a, b), (a, w), (a, x), (a, y), (a, z), (b, v), (b, x), (b, y), (b, z), (v, a), (v, w), (v, x), (v, z), (w, b), (w, x), (w, y), (x, y), (x, z), (y, v), (y, z), (z, w)}

the uncovered set $P_{(2)}$ is {a, b, v},

the union of minimal weakly stable sets $S_{(1)}$ is {a, b, v, y, w},

$P_{(3)}$ coincides with the whole set A: alternatives x and z are 3-stable, but don't belong to the union of minimal weakly stable sets.

Example 5. (see Figure 5)

A={a, b, v, w, x, y, z}

μ={(a, b), (a, w), (a, x), (a, y), (a, z), (b, v), (b, x), (b, y), (b, z), (v, a), (v, w), (v, x), (v, z), (w, b), (w, x), (w, y), (x, y), (x, z), (y, v), (z, w), (z, y)}

Example 6. (see Figure 6)

A={a, b, v, w, x, y, z}

μ={(a, b), (a, w), (a, x), (a, y), (a, z), (b, v), (b, x), (b, y), (b, z), (v, a), (v, x), (v, z), (w, b), (w, v), (w, x), (w, y), (x, y), (x, z), (y, v), (z, w), (z, y)}

Example 7. (See Figure 7) $\exists\mu$:

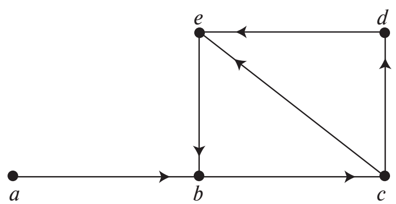

Fig. 7

$MU \subset MWS^{I}$; $MWS^{I} \backslash MWS^{II} \neq \varnothing$ & $MWS^{II} \backslash MWS^{I} \neq \varnothing$.

A={a, b, c, d, e}

μ={(a, b), (b, c), (c, d), (c, e), (d, e), (e, b)}

MU={a}; UC^{I}={a, c}; UC^{II}={a, c, b, e}; UC^{III}={a, c, d}; MWS^{I}={a, c, e}; MWS^{II}={a, b, c, d}; $MWS^{I} \backslash MWS^{II}$={e}; $MWS^{II} \backslash MWS^{I}$={b, d}; UT={a, c, d, e}; $UC^{IV}=UC^{V}=UCp=MD=A$

Appendix 2

Here some useful propositions are given, which yield an algorithm for calculating the classes of k-stable alternatives and the minimal dominant sets for tournaments.

Let $L^2(x)$ denote x's *lower contour set of the second degree*, which consists of all those y that belong to the lower contour sets of alternatives from the lower contour set of x but at the same time do not belong to x's lower contour set itself, $y \in L^2(x) \Leftrightarrow \exists z \in L(x)$: $y \in L(z)$ & $y \notin L(x)$. Correspondingly x's *lower contour set of the k'th degree* $L^k(x)$ consists of all those y that belong to the lower contour sets of alternatives from $L^{k-1}(x)$ but at the same time do not belong to any $L^i(x)$ of the degree less than k (k=1 included), $y \in L^k(x) \Leftrightarrow \exists z \in L^{k-1}(x)$: $y \in L(z)$ & $y \notin L^i(x)$ for all i: i<k. In some instances lower contour set of x will be referred to as lower contour set of the degree 1, and we put $L^0(x)=\{x\}$.

It follows from the definition that if $L^k(x) \neq \varnothing$ then $\forall i$: i<k $L^i(x) \neq \varnothing$. If $\exists y \in L^k(x)$ then there is a path $x \to y_1 \to y_2 \to \ldots \to y_{k-2} \to y_{k-1} \to y$ from x to y such that $\forall i$ $y_i \in L^i(x)$. It follows from the construction of $L^k(x)$ that if μ is a tournament and $y \in L^k(x)$ then y dominates x and all alternatives from $L^i(x)$ for all i: i<k-1. Indeed, if $y \in L^k(x)$ then $\forall i$: i<k $y \notin L^i(x)$, consequently $\forall z \in L^{i-1}(x)$ $y \notin L(z)$ $\Rightarrow \forall z \in L^{i-1}(x)$ for any i: i<k $y \in D(z)$, i.e. $y\mu z$.

Historical remark. Miller (1980, p. 70) introduced sets $R_k(x)$ similar to $L^k(x)$. In terms of $R_k(x)$ $L^k(x)=R_k(x) \backslash R_{k-1}(x)$.

Lemma 8. If $x \to y_1 \to y_2 \to \ldots \to y_{k-2} \to y_{k-1} \to y$ is a minimal path from x to y such that l(x, y)=k, then $y_i \in L^i(x)$ for all i: i<k & $y \in L^k(x)$.

Corollary. If y is reachable from x, then $\exists k$: $y \in L^k(x)$. Therefore, if x is generally stable then the union of all lower contour sets of x ($L^0(x)$ included) coincides with the whole set A, $\cup L^i(x)=A$.

According to Lemma 8 $x \in SP_{(k)}$ implies $\exists y \in L^k(x)$. Therefore, if x is k-stable (k>1) there is an alternative y that dominates x and all alternatives from all $L^i(x)$, i: i⩽k-2; $x \in SP_{(k)} \Rightarrow \exists y$: $y\mu x$ & $\forall z \in L^i(x)$ for all i: i⩽k-2 $y\mu z$. Alternatively, if there is y that dominates x and all alternatives from all $L^i(x)$, i: i⩽k-2, then the degree of stability of x is no less than k. Indeed, if there is an alternative y that dominates x and all alternatives from $L^i(x)$ for any i: i⩽k-2, then y is either not reachable from x or belongs to $L^j(x)$ for any j: j⩾k. In both cases y is not reachable from x in less than k steps, thus the degree of stability of x is no less than k.

If x is k-stable (k>1) then all its lower contour sets of the degree k and less are nonempty, but all lower contour sets of the degree greater than k are empty, $x \in SP_{(k)} \Rightarrow \forall i$: i⩽k $L^i(x) \neq \varnothing$ & $\forall j$: j>k $L^j(x)=\varnothing$. Indeed, by definition if x is k-stable then $L^k(x) \neq \varnothing$, therefore $\forall i$: i<k $L^i(x) \neq \varnothing$. Also by definition there

is no alternative in A not reachable from x in more than k steps, which means $\forall i$: $i>k$ $L^i(x)=\varnothing$.

This yields the following algorithm for calculating $\{SP_{(k)}\}$ and MD. To establish, which point-class any given alternative x belongs to, one should calculate, one by one, all lowers contours $L^i(x)$ of x. If $\cup L^i(x)=A$ for all i: $i\geq 0$, then x is generally stable, consequently it belongs to MD. If x is generally stable then its nonempty lower contour of the greatest degree ($L^k(x)$: $L^k(x)\neq\varnothing$ & $\forall j$: $j>k$ $L^j(x)=\varnothing$) will determine x's degree of stability. Repeating the calculation for all alternatives in any given A one obtains MD and $\{SP_{(k)}\}$.

An algorithm for calculation of set-classes $\{SS_{(k)}\}$ is related to calculation of weakly stable sets under Theorem 1 and will be presented in the next paper.

Appendix 3

Lemma 1. Proof: Let on the contrary $D_1\backslash D_2\neq\varnothing$ & $D_2\backslash D_1\neq\varnothing$. Then $\exists x, y$: $x\in D_1\backslash D_2$, $y\in D_2\backslash D_1$, $x\neq y$. Since D_1 is a dominant set, then by definition $x\mu y$. Similarly, since D_2 is a dominant set as well, $y\mu x$, a contradiction.

Lemma 2. Proof: Since A is finite, the number of subsets of A is finite. According to Lemma 1 one can order those subsets that are dominant sets by inclusion, and because their number is finite, one of them must be minimal.

If MD is a minimal dominant set then by definition there is no dominant set D such that D is a subset of MD ($D\subset MD$). Let D be a dominant set and $D\neq MD$. According to Lemma 1 either MD is a subset of D ($MD\subset D$) or D is a subset of MD ($D\subset MD$). The latter is not true according to the definition, which implies MD must be a subset of D. Hence there is no minimal dominant set other than MD, i.e. minimal dominant set is unique.

Lemma 3. Proof: By construction $\forall j\neq i$ $MD_{(j)}\cap MD_{(i)}$ and $\cup MD_{(i)}=A$, therefore $\forall x$: $x\in D\Rightarrow\exists i$: $x\in MD_{(i)}$. If $D\cap MD_{(i)}\neq\varnothing$, then $D\supset MD_{(i)}$. Suppose on the contrary $D\cap MD_{(i)}\neq\varnothing$ & $MD_{(i)}\backslash D\neq\varnothing$. Since D is a dominant set, $\forall x, y$: $x\in D\cap MD_{(i)}$, $y\in MD_{(i)}\backslash D\Rightarrow x\mu y$. By definition of $MD_{(i)}$ $\forall x, y$: $x\in D\cap MD_{(i)}$, $y\in(A\backslash\cup MD_{(j)})\backslash MD_{(i)}$, $1\leqslant j\leqslant i-1\Rightarrow x\mu y$. Therefore $D\cap MD_{(i)}$ is a dominant set in $A\backslash\cup MD_{(j)}$, $1\leqslant j\leqslant i-1$. $MD_{(i)}\backslash D\neq\varnothing\Rightarrow D\cap MD_{(i)}\subset MD_{(i)}$ (Lemma 1), therefore $MD_{(i)}$ is not a minimal dominant set in $A\backslash\cup MD_{(j)}$, $1\leqslant j\leqslant i-1$, a contradiction. Consequently, D must be a direct sum of a certain number of sets $MD_{(i)}$.

By construction $\forall x, y$: $x\in MD_{(i)}$, $y\in MD_{(k)}$ for any k and i: $k>i\Rightarrow x\mu y$. Therefore, if $MD_{(i)}$ is a minimal dominant set with the greatest degree among those included in D, all $MD_{(j)}$, $1\leqslant j<i-1$ must belong to D as well. Had it been otherwise, D would have not been a dominant set.

Lemma 4. Proof: If MWS is a minimal weakly stable set then there is no weakly stable set WS such that is a subset of MWS, $WS\subset MWS$. If MWS is not a subset of MD then $MWS\backslash MD\neq\varnothing$ and $MWS\cap MD\neq MWS$. By definition of MD, $\forall x, y$: $x\notin MWS\cap MD$ & $y\in MWS\cap MD$ & $x\mu y\Rightarrow x\in MD\backslash MWS$. By definition of a weakly stable set, $\exists z$: $z\in MWS$ & $z\mu x$. $x\in MD\Rightarrow z\in MD\Rightarrow z\in MWS\cap MD$. Thus ($\exists x, y$: $x\notin MSW\cap MD$ & $y\in MSW\cap MD$ & $x\mu y$) $\Rightarrow\exists z\in MSW\cap MD$: $z\mu x$. By definition $MSW\cap MD$ is a weakly stable set. $(MSW\cap MD)\subset MWS\Rightarrow MWS$ is not a minimal weakly stable set, a contradiction. Hence a minimal weakly stable set is a subset of the minimal dominant set.

Lemma 5. Proof: Sufficiency. B is weakly stable $\Leftrightarrow\forall x\notin B$: $B\cap L(x)\neq\varnothing\Rightarrow B\cap D(x)\neq\varnothing$. Therefore, if $\forall x\notin B\Rightarrow B\cap D(x)\neq\varnothing$ then B is weakly stable. Necessity. Since μ is a tournament, $\forall x\in A\Rightarrow D(x)\cup L(x)\cup\{x\}=A$. If $x\notin B$ then $B\subseteq(D(x)\cup L(x))$. Consequently, $B\cap(D(x)\cup L(x))=(B\cap D(x))\cup(B\cap L(x))=B\neq\varnothing$. Consequently, $\forall B$: $B\cap L(x)=\varnothing\Rightarrow B\cap D(x)\neq\varnothing$. Therefore, if B is weakly stable, i.e. if also $B\cap L(x)\neq\varnothing\Rightarrow B\cap D(x)\neq\varnothing$ then $\forall x\Rightarrow B\cap D(x)\neq\varnothing$.

Theorem 1. Proof: Suppose $x\in B$, B is a minimal weakly stable set. Then $B\backslash\{x\}$ is not weakly stable. According to Lemma 5 $\exists y\notin B\backslash\{x\}$: $(B\backslash\{x\})\cap D(y)=\varnothing$. At the same time $\forall y\notin B\Rightarrow B\cap D(y)\neq\varnothing$. Therefore either $x=y$ or $x\in D(y)$.

If $x=y$ then $(B\backslash\{x\})\cap D(y)=\varnothing\Leftrightarrow B\backslash\{x\}\subseteq A\backslash(D(x)\cup\{x\})\Leftrightarrow B\subseteq A\backslash D(x)$. If B is weakly stable then by Corollary of Lemma 5 $A\backslash D(x)$ must be weakly stable as well.

If $x\in D(y)$ ($\Leftrightarrow y\in L(x)$) then $B\backslash\{x\}\subseteq A\backslash(D(y)\cup\{y\})\Leftrightarrow B\subseteq(A\backslash(D(y)\cup\{y\}))\cup\{x\}$, $y\in L(x)$. If B is weakly stable then $(A\backslash(D(y)\cup\{y\}))\cup\{x\}$ must be weakly stable.

As a result, if B is a minimal weakly stable set and $x\in B$ then it is necessary that either one of the two conditions holds:

(*) $A\backslash D(x)$ is weakly stable;

(**) $\exists y\in L(x)$: $(A\backslash(D(y)\cup\{y\}))\cup\{x\}$ is weakly stable.

Let's prove that either one of the conditions is sufficient for the existence of a minimal weakly stable set B: $x\in B$.

Suppose (*) holds. If $A\backslash D(x)$ is minimal then $B=A\backslash D(x)$. If it is not, then $\exists C$: $C\subset A\backslash D(x)$, C is a minimal weakly stable set. Evidently, $x\notin A\backslash(D(x)\cup\{x\})$ & $(A\backslash(D(x)\cup\{x\}))\cap D(x)=\varnothing$, therefore $A\backslash(D(x)\cup\{x\})$ is not weakly stable. Since C is weakly stable, C is not a subset of $A\backslash(D(x)\cup\{x\})$. But $C\subset A\backslash D(x)$, therefore $x\in C$ and $B=C$.

Suppose (**) holds. If $(A\backslash(D(y)\cup\{y\}))\cup\{x\}$ is minimal then $B=(A\backslash(D(y)\cup\{y\}))\cup\{x\}$. If it is not, then $\exists C$: $C\subset(A\backslash(D(y)\cup\{y\}))\cup\{x\}$, C is a minimal weakly stable set. Evidently, $y\notin A\backslash(D(y)\cup\{y\})$ and $A\backslash(D(y)\cup\{y\})\cap D(y)=\varnothing$, therefore $A\backslash(D(y)\cup\{y\})$ is not weakly stable. Since C is weakly stable, C is not a subset of $A\backslash(D(y)\cup\{y\})$. But $C\subset(A\backslash(D(y)\cup\{y\}))\cup\{x\}$, therefore $x\in C$ and $B=C$.

Thus, x belongs to any minimal weakly stable set iff $A\backslash D(x)$ is a weakly stable set or $\exists y\in L(x)$ such that $(A\backslash(D(y)\cup\{y\}))\cup\{x\}$ is a weakly stable set.

Suppose $A\setminus D(x)$ is not weakly stable then $\exists z$: $z\notin A\setminus D(x)$ and $(A\setminus D(x))\cap D(z)=\varnothing$, then $A\setminus D(x)\subseteq A\setminus(D(z)\cup\{z\})$, then $D(z)\cup\{z\}\subseteq D(x) \Leftrightarrow$ ($z\mu x$ & $D(z)\subseteq D(x)$). Therefore, if $A\setminus D(x)$ is not weakly stable then x is covered according to the third definition of covering.

Suppose x is covered by z, $\exists z$: $z\mu x$ & $D(z)\subseteq D(x)$. Then $A\setminus D(x)\subseteq A\setminus(D(z)\cup\{z\})$, therefore $\exists z$: $z\notin A\setminus D(x)$ & $(A\setminus D(x))\cap D(z)=\varnothing$. According to Lemma 5 $A\setminus D(x)$ is not weakly stable.

Therefore $A\setminus D(x)$ is weakly stable iff x is uncovered according to the third definition of covering.

Suppose $\exists y\in L(x)$: $(A\setminus(D(y)\cup\{y\}))\cup\{x\}$ is not weakly stable then $\exists z$: $z\notin(A\setminus(D(y)\cup\{y\}))\cup\{x\}$ and $((A\setminus(D(y)\cup\{y\}))\cup\{x\})\cap D(z)=\varnothing$. $z\notin(A\setminus(D(y)\cup\{y\}))\cup\{x\} \Rightarrow (z\neq x$ & $z\in D(y)\cup\{y\})$. Since $\{x\}\cap D(y)=\{x\}$, $((A\setminus(D(y)\cup\{y\}))\cup\{x\})\cap D(z)=\varnothing \Rightarrow z\neq y$. Consequently, $z\in D(y)$ & $(A\setminus(D(y)\cup\{y\}))\cup\{x\}\subseteq A\setminus D(z)$, therefore $D(z)\cup\{z\}\subseteq D(y) \Leftrightarrow z\mu y$ & $D(z)\subseteq D(x)$. Consequently, if $\exists y\in L(x)$: $(A\setminus(D(y)\cup\{y\}))\cup\{x\}$ is not weakly stable then y is covered according to the third definition of covering.

Suppose $\exists y\in L(x)$: $(A\setminus(D(y)\cup\{y\}))\cup\{x\}$ is weakly stable. If y is uncovered then it is an uncovered alternative in L(x). If y is covered by some z, then $D(z)\cup\{z\}\subseteq D(y)$. By Lemma 5 $((A\setminus(D(y)\cup\{y\}))\cup\{x\})\cap D(z)\neq\varnothing$, consequently $x\in D(z)$, that is $z\in L(x)$. If z is uncovered then it is an uncovered alternative in L(x). Suppose z is covered. Then $D(z)\cup\{z\}\subseteq D(y) \Rightarrow (A\setminus(D(y)\cup\{y\})\cup\{x\}\subseteq(A\setminus(D(z)\cup\{z\})\cup\{x\}$. According to Corollary of Lemma 5 $(A\setminus(D(z)\cup\{z\})\cup\{x\}$ is weakly stable. After substituting z for y and repeating the logical steps listed above, one obtains that any alternative that covers z must belong to L(x) as well. L(x) is finite, therefore, since relation of covering is asymmetric and transitive, there must be at least one uncovered alternative in L(x).

Therefore $\forall y\in L(x)$ a set $(A\setminus(D(y)\cup\{y\}))\cup\{x\}$ is not weakly stable iff all alternatives in L(x) are covered, and $\exists y\in L(x)$: $(A\setminus(D(y)\cup\{y\}))\cup\{x\}$ is weakly stable iff there is an uncovered alternative in L(x).

As a result, x belongs to a minimal stable set iff x is uncovered (according to the third definition of covering) or some alternative from the lower contour of x is uncovered (according to the third definition of covering).

Corollary to Theorem 1a. Proof: $UC^{III}\subseteq MWS^{II}$ directly follows from the proposition of the Theorem. $MWS^{II}\subseteq UCp$ follows from the proposition of the Theorem, definition of UCp and $UC^{III}\subseteq UC^{IV}$.

Lemma 6. Proof:

In tournaments $UC^{I}=UC^{II}=UC^{III}=UC^{IV}=UC^{V}\subseteq MWS^{I}$ (Corollary to Theorem 1) while $MWS^{I}\subset UC^{I}$, $MWS^{I}\subset UC^{II}$, $MWS^{I}\subset UC^{III}$, $MWS^{I}\subset UC^{IV}$, $MWS^{I}\subset UC^{V}$ (Example 1) are also possible.

It is possible that $MWS^{I}\setminus MWS^{II}\neq\varnothing$ & $MWS^{II}\setminus MWS^{I}\neq\varnothing$ (Example 7).

In tournaments $MWS^{I}\subseteq MU=MD$ (Lemma 4), while $MU\subset MWS^{I}$ (Example 7) is also possible.

Let's prove $MWS^{I}\subseteq UCp$. Suppose alternative x belongs to a minimal weakly stable (according to the first definition) set B, $x\in B$. Then $B\setminus\{x\}$ is not weakly stable. Consequently, $\exists y\notin B\setminus\{x\}$: $(B\setminus\{x\})\subseteq(L(y)\cup H(y))$ & $(B\setminus\{x\})\cap L(y)\neq\varnothing$.

1) Suppose $y=x$. Then $\forall z$: $z\mu x \Rightarrow z\notin B$. Since B is a weakly stable set, $\forall z$: $z\mu x \Rightarrow \exists w\in B$: $w\mu z$. ($w\in B$ & $w\mu z$) $\Rightarrow w\neq x \Rightarrow w\in B\setminus\{x\} \Rightarrow w\in L(x)\cup H(x)$. Consequently if $y=x$, then $\forall z$: $z\mu x \Rightarrow \exists w$: ($x\mu w$ & $w\mu z$) or ($x\tau w$ & $w\mu z$), i.e. $x\in UC^{III}$. Therefore, if x is uncovered according to the third definition of covering, $x\notin UC^{III}$, and B is a weakly stable (according to the first definition) set then $y\neq x$.

2) Suppose $y\neq x$. Suppose x is not covered according to the second definition of covering, $x\notin UC^{II}$. Since B is a weakly stable set and $(B\setminus\{x\})\cap L(y)\neq\varnothing$ there must be an alternative w, $w\in B$: $w\mu y \Leftrightarrow w\in D(x)$. $(B\setminus\{x\})\subseteq(L(y)\cup H(y)) \Rightarrow w=x \Leftrightarrow y\in L(x)$. ($x\mu y$ & $x\notin UC^{II}$) $\Rightarrow \exists z$: $z\mu x$ & $z\mu y$. ($z\mu x$ & $z\mu y$ & $(B\setminus\{x\})\subseteq(L(y)\cup H(y))$) $\Rightarrow z\notin B$. Since B is a weakly stable set, $z\mu x$ and $z\notin B$ imply there must be v, $v\in B\setminus\{x\}$: $v\mu z$. (($\exists v\in B\setminus\{x\}$: $v\mu z$) & $(B\setminus\{x\})\subseteq(L(y)\cup H(y))$) $\Leftrightarrow$ $(L(y)\cup H(y))\cap D(z)\neq\varnothing \Leftrightarrow$ y is not covered by z according to the third definition of covering, $y\in UC^{III}$.

Therefore, if there is an alternative z, which 1) covers an alternative x according to the fourth definition of covering (i.e. according to the second and third definitions simultaneously) and 2) also covers all alternatives from the lower contour set of x according to the third definition of covering then x does not belong to any minimal weakly stable set. Thus $MWS^{I}\subseteq UCp$. $\exists\mu$: $MWS^{I}\subset UCp$ is proved by Example 7.

Let's prove $MWS^{I}\subseteq UT$. Suppose alternative x belongs to a minimal weakly stable (according to the first definition) set B, $x\in B$. Suppose not all alternatives in B are reachable from x. Let C denote a set of all those alternatives in B, which are not reachable from x. By supposition $C\subset B$ & $C\neq\varnothing$. If $y\in C$ & $z\in B\setminus C$ then either $y\mu z$ or $y\tau z$, otherwise there would be a path from x to y through z. Therefore if $\forall y\in C$ $\exists w$: $w\mu y$ then $w\notin B\setminus C$. Suppose $\exists w$, y, $w\in A\setminus B$, $y\in C$: $w\mu y$. Since B is a weakly stable (according to the first definition) set then $\exists z$, $z\in B$: $z\mu w$. $z\notin B\setminus C$, otherwise there would be a path from x to y through z and w, consequently $z\in C$. Therefore, since none of the alternatives from $B\setminus C$ dominates any alternative in C, ($\exists w$, y, $w\in A\setminus C$, $y\in C$: $w\mu y$) $\Rightarrow$ ($\exists z$, $z\in C$: $z\mu w$). Therefore C is a weakly stable set according to the first definition. Since $C\subset B$, B is not a minimal weakly stable set, a contradiction. Therefore $C=\varnothing$ and any alternative in a minimal weakly stable set MWS^{I} is reachable from any over alternative from this set. Suppose x belongs to a minimal weakly stable (according to the first definition) set B, $x\in B$, and y dominates x, $y\mu x$. If $y\in B$ then y is reachable from x. If $y\notin B$ then, since B is a weakly stable set, $\exists z\in B$: $z\mu y$. Since

$z \in B$, z is reachable from x, consequently, y is reachable from x trough z. Thus if an alternative x belongs to a minimal weakly stable set MWS^{I}, any alternative, which dominates x, is reachable from x, i.e. x is untrapped. $\exists\mu$: $MWS^{I} \subset UT$ is proved by Example 7.

In tournaments $UC^{II}=UC^{IV}=UC^{V} \subseteq MWS^{II} \subseteq MU=UT$ while $MU \subset MWS^{II}$; $UT \subset MWS^{II}$ (Example 1) and $MWS^{II} \subset UC^{II}$, $MWS^{II} \subset UC^{IV}$, $MWS^{II} \subset UC^{V}$ (Example 2) are also possible.

$UC^{I} \subseteq UC^{III}$ & $UC^{III} \subseteq MWS^{II}$ (Corollary to Theorem 1a) $\Rightarrow UC^{I} \subseteq MWS^{II}$.

$MWS^{II} \subseteq UCp$ & $UCp \subseteq MD \Rightarrow MWS^{II} \subseteq MD$.

Lemma 7. If x is k-stable and y is (k+n)-stable, n: $n \geqslant 2$, then x dominates y, $x \in SP_{(k)}$ & $y \in SP_{(k+n)}$, n: $n \geqslant 2$, $\Rightarrow x\mu y$.

Proof: If x is k-stable, each alternative in A is reachable from x in no more than k steps. Suppose $y\mu x$, then each alternative in A is reachable from y in no more than k+1 steps through x, therefore the degree of stability of y is no greater than k+1, a contradiction.

Theorem 2. Proof: If m=2 then the proposition is evident.

If m>2 then by induction.

1) If $m=\max_{x \in MD} l_{\max}(x)$ then $SP_{(m)} \neq \varnothing$.

2) Suppose $SP_{(i+1)}$ is nonempty, let's prove $SP_{(i)}$, $i \geqslant 3$ is nonempty too, $SP_{(i+1)} \neq \varnothing \Rightarrow SP_{(i)} \neq \varnothing$, $i \geqslant 3$.

Let $S=SP_{(m)}+\ldots+SP_{(i+1)}$, then $MD \backslash S=SP_{(1)}+SP_{(2)}+\ldots+SP_{(i)}$. If an alternative belongs to $MD \backslash S$, its degree of stability is no greater than i, if it belongs to S — no less than i+1.

$SP_{(2)}$ is nonempty as it is the uncovered set, $SP_{(2)} \neq \varnothing$. By supposition $SP_{(i+1)} \neq \varnothing$. Therefore both S and $MD \backslash S$ are nonempty proper subsets of MD, $S \subset MD$, $MD \backslash S \subset MD$.

Since MD is a minimal dominant set, $\exists x, y$, $x \in S$, $y \in MD \backslash S$: $x\mu y$. Had it been otherwise all alternatives from $MD \backslash S$ would have dominated all alternatives from S, which would have meant $MD \backslash S$ is a dominant set, and, respectively, MD is not a minimal one.

The degree of stability of y is no greater than i, the degree of stability of x is no less than i+1. According to Lemma 7, $x\mu y \Rightarrow x \in SP_{(i+1)}$ & $y \in SP_{(i)}$, therefore $SP_{(i)} \neq \varnothing$. The induction stops when it comes to i=3.

Theorem 3. Proof: 1) $P_{(2)}=UC \subseteq MWS=S_{(1)} \subseteq UCp=P_{(3)}$ is a proposition of the Corollary to Theorem 1. Example 4 proves that $\exists\mu$: $P_{(2)} \subset S_{(1)} \subset P_{(3)}$.

2) Suppose x is k-stable, $x \in SP_{(k)}$. There is always a minimal k-stable set B, which consists of only one alternative — x, B={x}. Therefore, if x belongs to $P_{(k)}=SP_{(2)}+\ldots+SP_{(k)}$ then it must belong to $S_{(k)}=SS_{(1)}+SS_{(2)}+\ldots+SS_{(k)}$ as well, which implies $P_{(k)} \subseteq S_{(k)}$ and $x \in SP_{(k)} \Rightarrow x \in SS_{(i)}$, $i \leqslant k$.

Suppose x belongs to a minimal k-stable set B, k>1. Since B is k-stable, it is always possible to reach any alternative outside B from some alternative in B in no more than k steps.

If all alternatives in $B \backslash \{x\}$ are dominated by x, it is possible to reach any alternative in B from x in 1 step. Therefore it is possible to reach from x any other alternative in no more k+1 steps, i.e. $x \in P_{(k+1)}$, which implies $x \in P_{(k+2)}$.

Suppose $\exists y \in B$: $y\mu x$. Since B is minimal, then $B \backslash \{x\}$ is not k-stable, which implies that there is z outside $B \backslash \{x\}$, not reachable from any alternative from $B \backslash \{x\}$ (y included) in less than k+1 steps. $\forall w \in B \backslash \{x\} \Rightarrow z\mu w \Rightarrow z \neq x$, since by supposition $\exists y \in B$: $y\mu x$. z is reachable from x in no less than k steps, otherwise there would be a path from y to z through x of length less than k+1. But since B is a k-stable set, there must be an alternative in B, from which z is reachable in no more than k steps. Since it can't be any alternative from $B \backslash \{x\}$, it must be x. Therefore z is reachable from x in exactly k steps. Let $x \to z_1 \to z_2 \to \ldots \to z_{k-1} \to z$ be a path of length k from x to z. The second alternative in the sequence z_1 must dominate y: $z_1\mu y$, otherwise there would be a path $y \to z_1 \to z_2 \to \ldots \to z_{k-1} \to z$ from y to z of length k, i.e. of length less than k+1. $x\mu z_1$ & $z_1\mu y$ & $y\mu x \Rightarrow$ y is reachable from x in 2 steps. Thus any alternative in $B \backslash \{x\}$ is either dominated by x or reachable from x in 2 steps. Consequently, since B is k-stable, any alternative in A is reachable from x in no more than k+2 steps, i.e. $x \in P_{(k+2)}$.

Thus $x \in S_{(k)} \Rightarrow x \in P_{(k+2)}$, which implies $S_{(k)} \subseteq P_{(k+2)}$.

Corollary 2. Proof: $x \in SS_{(2)} \Rightarrow x \in S_{(2)} \Rightarrow x \in P_{(4)}=SP_{(2)}+SP_{(3)}+SP_{(4)} \Rightarrow \forall k$: k=2 or k>4 $x \notin SP_{(k)}$. $x \notin SP_{(2)}$ because $x \in SP_{(2)} \Rightarrow x \in SS_{(1)}$ (according to Theorem 3).

Corollary 3. Proof: $x \in SS_{(k)} \Rightarrow x \notin S_{(k-1)} \Rightarrow x \notin P_{(k-1)}=SP_{(2)}+\ldots+SP_{(k-1)} \Rightarrow \forall i$: i<k $x \notin SP_{(i)}$. $x \in SS_{(k)} \Rightarrow x \in S_{(k)} \Rightarrow x \in P_{(k+2)}=SP_{(2)}+\ldots+SP_{(k+2)} \Rightarrow \forall i$: i>k+2 $x \notin SP_{(i)}$.

Lemma 8. Proof: By induction.

$x\mu y_1 \Rightarrow y_1 \in L(x)$

Suppose $\forall j$: $j \leqslant i-1$ $y_j \in L^j(x)$. Suppose $\exists z \in L^j(x)$: $z\mu y_i$ for all j: $0 \leqslant j < i-1$. Since $z \in L^j(x)$, there must be a path from x to z: $x \to z_1 \to z_2 \to \ldots \to z_{j-1} \to z$: $z_m \in L^m(x)$ for all m: $m \leqslant j-1$. The length of this path is j, $0 \leqslant j < i-1$. Since $z\mu y_i$, there is a path from x to y $x \to z_1 \to z_2 \to \ldots \to z_{j-1} \to z \to y_i \to y_{i+1} \to \ldots \to y_{k-1} \to y$. The length of this path is $j+(k-(i-1))=k+(j-(i-1))<k$, i.e. this path is shorter than the path $x \to y_1 \to y_2 \to \ldots \to y_{k-2} \to y_{k-1} \to y$, a contradiction. Therefore $\forall z \in L^j(x)$, $\forall j$: $0 \leqslant j < i-1 \Rightarrow y_i\mu z$. According to the definition of $L^j(x)$ $y_i \notin L^j(x)$ for all j: $1 \leqslant j < i$. At the same time $\exists y_{i-1} \in L^{i-1}(x)$: $y_{i-1}\mu y_i$. According to the same definition $y_i \in L^i(x)$.

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Препринт WP7/2008/03
Серия WP7
Теория и практика общественного выбора

А.Н. Субочев

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ЛР № 020832 от 15 октября 1993 г.
Отпечатано в типографии ГУ ВШЭ с представленного оригинал-макета.
Формат 60×84 $^1/_{16}$. Бумага офсетная. Тираж 150 экз. Уч.-изд. л. 1,8.
Усл. печ. 1,9. Заказ № . Изд. № 873

ГУ ВШЭ. 125319, Москва, Кочновский проезд, 3
Типография ГУ ВШЭ. 125319, Москва, Кочновский проезд, 3
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